Hawkes Processes with Stochastic Excitations - *Supplemental Materials*

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A. Simulation

This proof is adapted from Dassios and Zhao (2013) to our setting. We first note that the intensity λ_t increases by Y_j at jump time T_j . That is, the right limit $\lambda_{T_j^+}$ is equal to the left limit $\lambda_{T_j^-}$ plus Y_j . Thus, after the j-th jump, the intensity λ_t would follow

$$\lambda_t = \big(\lambda_{T_j^+} - a\big)\, e^{-\delta(t-T_j)} + a, \quad T_j \le t < T_j + S_{j+1},$$

where T_j and $\lambda_{T_j^+}$ are observed at the j-th jump. Here S_{j+1} is the inter-arrival time for the $(j+1)$-th jump:

$$S_{j+1} = T_{j+1} - T_j \, .$$

Given the intensity function, we can derive the cumulative density function for S_{j+1} as

$$F_{S_{j+1}}(s) = 1 - \exp\left(- \big(\lambda_{T_j^+} - a\big)\, \frac{1-e^{-\delta s}}{\delta} - as \right) \quad .$$

Note that we can decompose S_{j+1} into two simpler and independent random variables $S^{(1)}_{j+1}$ and $S^{(2)}_{j+1}$:

$$\begin{aligned}
\mathbb{P}(S_{j+1} > s) &= \exp\left(- \big(\lambda_{T_j^+} - a\big)\, \frac{1-e^{-\delta s}}{\delta} \right) \times e^{-as} \\
&= \mathbb{P}\Big(S^{(1)}_{j+1} > s\Big) \times \mathbb{P}\Big(S^{(2)}_{j+1} > s\Big) \\
&= \mathbb{P}\Big(\min\big(S^{(1)}_{j+1}, S^{(2)}_{j+1}\big) > s\Big).
\end{aligned}$$

We have define

$$\begin{aligned}
F_{S^{(1)}_{j+1}}(s) &= \mathbb{P}\Big(S^{(1)}_{j+1} \le s\Big) = 1 - \exp\left(- \big(\lambda_{T_j^+} - a\big)\, \frac{1-e^{-\delta s}}{\delta} \right), \\
F_{S^{(2)}_{j+1}}(s) &= \mathbb{P}\Big(S^{(2)}_{j+1} \le s\Big) = 1 - e^{-as}.
\end{aligned}$$

for $0 \le s < \infty$. Note that $S^{(1)}_{j+1}$ is a defective random variable (Dassios and Zhao, 2013).

Now, to simulate S_{j+1}, we simply need to independently simulate both $S^{(1)}_{j+1}$ and $S^{(2)}_{j+1}$. Simulating $S^{(2)}_{j+1}$ is trivial since $S^{(2)}_{j+1}$ follows an exponential distribution with rate parameter a. To simulate $S^{(1)}_{j+1}$, we use the inverse CDF approach:

$$S^{*}_{j+1} = -\frac{1}{\delta} \ln\left(1 + \frac{\delta \ln(v)}{\lambda_{T_j^+} - a} \right) \qquad \text{if } \exp\left(- \frac{\lambda_{T_j^+} - a}{\delta} \right) \le v < 1,$$

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we discard S^*_{j+1} otherwise, that is, $v < \exp\left(-\frac{\lambda_{T_j^+}-a}{\delta}\right)$ (this corresponds to the defective part), where v is simulated from a standard uniform distribution $V \sim U(0,1)$.

The following Algorithm 1 simulates the Geometric Brownian Motion and the exponential Langevin dynamics, both are crucial elements for Stochastic Hawkes:

Algorithm 1 Simulation of Stochastic Processes for each Y_i

1. Given $Y_{i\text{–}1}$ and $\{T_{i\text{–}1}, T_i\}$
2. If $Y \sim$ Geometric Brownian Motion, then
   (a) Sample Y_i through

$$u \sim N(0, \sigma^2(T_i - T_{i-1})), \quad Y_i = Y_{i-1}\exp\left(\mu(T_i - T_{i-1}) + u\right)$$

   If $Y \sim$ Exponential Langevin, then
   (a) Sample Y_i using

$$u \sim N\Big(0, \frac{\sigma^2}{2k}\big(1 - e^{-2k(T_i - T_{i-1})}\big)\Big), \quad Y_i = \exp\Big(\log Y_{i-1} e^{-k(T_i - T_{i-1})} + \mu(1 - e^{-k(T_i - T_{i-1})}) + u\Big)$$

B. Likelihood function

The likelihood derivation can be decomposed into these basic parts:

- For the occurrence of the first event T_1, note that the distribution of T_1: $P(T_1 \le t) = 1 - P(T_1 > t) = 1 - P(N_t - N_0 = 0) = 1 - \exp(-\int_0^t \lambda_v \, dv)$. Since T_1 is the time of the first event, this event must be an immigrant, thus $Z_{10} = 1$. We get

$$f_{T_1}(t) = \big(a + (\lambda_0 - a)e^{-\delta t}\big)\exp\left(-\int_0^t a + (\lambda_0 - a)e^{-\delta v} \, dv\right).$$

- For the occurrence of the second event T_2, we get by similar calculations $P(T_2 \le t \,|\, T_1) = 1 - P(T_2 > t \,|\, T_1) = 1 - P(N_t - N_{T_1} = 0) = 1 - \exp(-\int_{T_1}^t \lambda_v \, dv)$. Depending on whether this event is an immigrant or an offspring (from the first event), we can write

$$f_{T_2}(t) = \left[\big(a + (\lambda_0 - a)e^{-\delta t}\big)\exp\left(-\int_{T_1}^t a + (\lambda_0 - a)e^{-\delta v} \, dv\right)\exp\left(-\int_{T_1}^t Y_1 e^{-\delta(v - T_1)} \, dv\right)\right]^{Z_{20}}$$
$$\cdot \left[Y_1 \exp(-\delta(t - T_1))\exp\left(-\int_{T_1}^t Y_1 e^{-\delta(v - T_1)} \, dv\right)\exp\left(-\int_{T_1}^t a + (\lambda_0 - a)e^{-\delta v} \, dv\right)\right]^{Z_{21}}.$$

  Notice that λ factorizes and following from equation and we get

$$f_{T_2}(t) = (a + (\lambda_0 - a)e^{-\delta t})^{Z_{20}}(Y_1 \exp(-\delta(t - T_1)))^{Z_{21}} \exp\left(-\int_{T_1}^t \lambda_v \, dv\right).$$

Iterating in similar fashion, we arrive at the likelihood function for Stochastic Hawkes.

C. Itô's Processes — Stochastic Differential Equations

We give a brief review on the Itô's process and subsequently Itô's formula which we used to derive the solutions to the Geometric Brownian Motion and exponential Langevin to facilitate simulation and inference algorithms.

$B_t := B(t)$ represents the 1-d Brownian motion, or otherwise known as Wiener process. It has the following properties:

- The mapping $t \mapsto B_t$ is continuous with $B_0 := 0$.
- Fix $T < \infty$. Meshing $[0, T]$ as $0 < T_0 < T_1 < ... < T_N = T$, the increments $B_{T_i} - B_{T_{i-1}}$ are independent for $i = 1, 2, ..., N$.
- For all $s < t$, the increment $B_t - B_s$ has a normal distribution with mean 0 and variance $t - s$.

The stochastic process $Y \equiv \{Y_t\}_{t \geq 0}$ that solves

$$Y_t = Y_0 + \int_0^t \mu(s, Y_s)\, ds + \int_0^t \kappa(s, Y_s)\, dB_s \tag{1}$$

is known as the Itô process. The functionals μ and κ are the drift and volatility function, respectively. The stochastic differential equation equivalent of the Itô's process is

$$dY_t = \mu(t, Y_t)\, dt + \kappa(t, Y_t)\, dB_t\, . \tag{2}$$

The Itô's formula is an essential tool used to find the differential of a time-dependent function of a stochastic process. Intuitively, the Ito formula corresponds to a chain rule in the stochastic process. To explain this, we first explain the case of the ordinary differential equation

$$\frac{d}{dt} Y_t = \mu(t, Y_t). \tag{3}$$

Let q be a function of Y_t, by Leibnitz chain rule, we get

$$\frac{d}{dt} q(t, Y_t) = \mu(t, Y_t) \frac{\partial}{\partial Y} q(t, Y_t), \quad t \geq 0.$$

Defining the linear operator $\check{\mathcal{T}} = \mu \partial_Y$, we get $dq(t, Y_t) = \check{\mathcal{T}} q(t, Y_t)\, dt$ where $\check{\mathcal{T}} q(t, Y_t) = \mu(t, Y_t)\, \partial_Y q(t, Y_t)$. We have the following Itô's Formula, see also Kloeden and Platen (1999):

Theorem 1 *Let Y_t satisfies the stochastic differential equation*

$$Y_t = Y_0 + \int_0^t \mu(s, Y_s)\, ds + \int_0^t \kappa(s, Y_s)\, dB_s\, . \tag{4}$$

Let $q(t, Y_t) \in C^2_\infty[(0, T) \times \mathbb{R}]$. Then q satisfies the stochastic differential equation

$$dq(t, Y_t) = \check{\mathcal{T}}_{t,x} q(t, Y_t)\, dt + \check{\mathcal{T}}_{x,x} q(t, Y_t)\, dB_t\, . \tag{5}$$

where $\check{\mathcal{T}}_{t,x}$ and $\check{\mathcal{T}}_{x,x}$ are defined by

$$\begin{aligned} \check{\mathcal{T}}_{t,x} &= \partial_t + \mu \partial_Y + \frac{1}{2} \kappa^2 \partial_Y^2\, , \\ \check{\mathcal{T}}_{x,x} &= \kappa \partial_Y\, . \end{aligned}$$

Invoking Itô's formula to the SDEs satisfied by Geometric Brownian Motion and exponential Langevin yields explicit formula for Y as presented in Section 3.

D. The Acceptance Probabilities for Metropolis Hastings Algorithms

The following table complements the remaining Acceptance Probabilities that we mentioned in the main text.

Table 1. Acceptance Probabilities for Metropolis Hastings Algorithms.

VARIABLE	ACCEPTANCE PROBABILITY
λ_0	$A\big(\lambda_0'\big) = \prod_{i=1}^{N(T)} \big(\frac{a+(\lambda_0'-a)e^{-\delta T_i}}{a+(\lambda_0-a)e^{-\delta T_i}}\big)^{Z_{i0}} \big(\frac{\lambda_0'}{\lambda_0}\big)^{\alpha_{Y_0}-1} \exp\big(-\big(\lambda_0'-\lambda_0\big)\big(\frac{1}{\delta}\big(1-e^{-\delta T}\big)+\beta_{\lambda_0}\big)\big)$
δ	$A\big(\delta'\big) = \prod_{i=1}^{N(T)} \big(\frac{a+(\lambda_0-a)\,e^{-\delta' T_i}}{a+(\lambda_0-a)\,e^{-\delta T_i}}\big)^{Z_{i0}} \big(\frac{\delta'}{\delta}\big)^{\alpha_\delta-1} \exp\big\{-\big(\delta'-\delta\big)\big[\sum_{i=1}^{N(T)}\sum_{j<i} Z_{ij}(T_i - T_j)\big] - (\lambda_0-a)\big[\frac{1}{\delta'}\big(1-e^{-\delta' T}\big)-\frac{1}{\delta}\big(1-e^{-\delta T}\big)\big] - \sum_{i=1}^{N(T)} Y_i\big[\frac{1}{\delta'}\big(1-e^{-\delta'(T-T_i)}\big)-\frac{1}{\delta}\big(1-e^{-\delta(T-T_i)}\big)\big] - \big(\delta'-\delta\big)\beta_\delta\big\}$
Y_i - EXPONENTIAL LANGEVIN	$A(Y_i') = \exp\big[-\frac{1}{\delta}(Y_i'-Y_i)(1-e^{-\delta(T-T_i)}) - \frac{k}{\sigma^2\phi_i^-\,\phi_i^+}\big\{\big(\log\big(\frac{Y_i'}{Y_{i-1}\phi_i}\big)-\mu\xi_i\phi_i^-\big)^2 - \big(\log\big(\frac{Y_i}{Y_{i-1}\phi_i}\big)-\mu\xi_i\phi_i^-\big)^2\big\} - \frac{k}{\sigma^2\phi_i^-\,\phi_i^+}\big\{\big(\log\big(\frac{Y_{i+1}}{Y_i'\phi_{i+1}}\big)-\mu\xi_{i+1}\phi_{i+1}^-\big)^2 - \big(\log\big(\frac{Y_{i+1}}{Y_i\phi_{i+1}}\big)-\mu\xi_{i+1}\phi_{i+1}^-\big)^2\big\}\big]\cdot\big(\frac{Y_i'}{Y_i}\big)^{\sum_{r=i+1}^{N_T} Z_{ri}-1}$

References

Dassios, A. and Zhao, H. (2013). Exact simulation of Hawkes process with exponentially decaying intensity. *Electronic Communications in Probability*, 18:1–13.

Kloeden, P. E. and Platen, E. (1999). *Numerical solution of stochastic differential equations*. Applications of Mathematics. Springer, Berlin, New York.